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# Fuzzy $n$ -Normed Space and Fuzzy $n$ -Inner Product Space

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## Abstract

In this paper we construct a new concept of fuzzy  $n$ -normed space and fuzzy  $n$ -inner product space. First, we show that every  $n$ -normed space can be constructed as a fuzzy  $n$ -normed space. Then we show that every  $n$ -normed space is a fuzzy  $n$ -normed space and every fuzzy  $n$ -normed space can also be reduced to a fuzzy  $(n-1)$ -normed space. Furthermore, for every  $n$ -inner product space can be constructed a fuzzy  $n$ -inner product space. Finally we show that every  $n$ -inner product space is a fuzzy  $n$ -inner product space and every fuzzy  $n$ -inner product space is a fuzzy  $n$ -normed space.

**Keywords:** Fuzzy  $n$ -normed space, fuzzy  $(n-1)$ -normed space, fuzzy  $n$ -inner product space

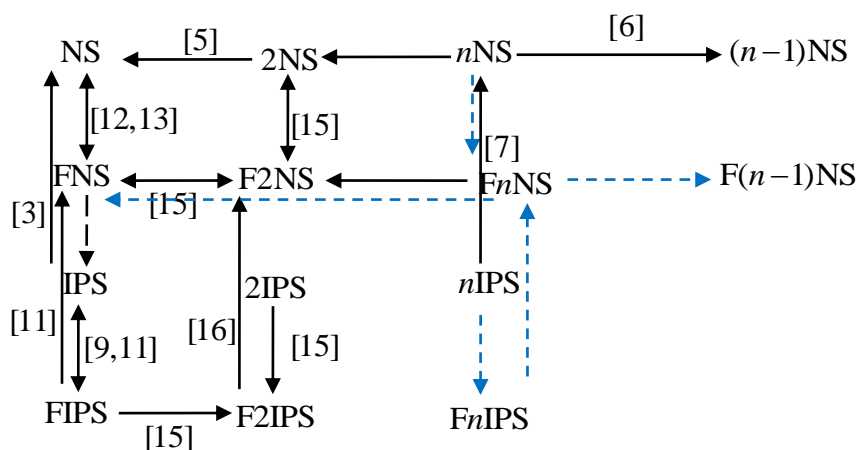
## 1. INTRODUCTION

Fuzzy set theory was first introduced by L. Zadeh on 1965. This theory continues to evolve in various disciplines. In mathematics, especially mathematical analysis, fuzzy concept has been developed in fuzzy  $n$ -normed space and fuzzy  $n$ -inner product.

Until now, it has been widely found outcomes related to fuzzy normed space and fuzzy inner product space. It has been proven in [6] that every  $n$ -normed space is a normed space and in [13] and [12] has also been demonstrated that the normed space and fuzzy normed space is equivalent so that every  $n$ -normed space is certainly a fuzzy normed space. On the other hand, in [15] it has also been demonstrated that the 2-normed spaces and fuzzy 2-normed spaces are also equivalent. However, unexplored in general how are the relationship between  $n$ -normed space with fuzzy

$n$ -normed space and fuzzy  $n$ -normed space with fuzzy normed space. Moreover, [7] has proved that every  $n$ -inner product space is an  $n$ -normed space. And [11] has also proved that the inner product space and fuzzy inner product space are equivalent. Moreover, it also has been proven in [15] that every 2-inner product space is a space fuzzy 2-inner product. However, unexplored in general how are the relationship between the  $n$ -inner product space and fuzzy  $n$ - inner product space and fuzzy  $n$ -normed space.

In this paper, first we give a definition of fuzzy  $n$ -normed and fuzzy  $n$ -inner product that are different from definition of fuzzy  $n$ -normed in [1] and fuzzy  $n$ -inner product in [17] which is also different from the definition put forward by other authors. Then some theorems are given to shows the relationship among  $n$ -normed space, fuzzy  $n$ -normed space, fuzzy normed space,  $n$ -inner product and fuzzy  $n$ -inner product space. For more details, one can see a chart in the Figure 1. (Blue arrows show the new theorem in this paper)



Notes:  $nNS$ :  $n$ -Norm Space;  $FnNS$ : Fuzzy  $n$ -Normed Space;  
 $nIPS$ :  $n$ -Inner Product Space;  $FnIPS$  : Fuzzy  $n$ -Inner Product Space.

**Figure 1.** The relationship among  $n$ -normed space, fuzzy  $n$ -normed space,  $n$ -inner product space and fuzzy  $n$ -inner product space

## 2. DISCUSSION

Let  $X$  be any nonempty set. Then a fuzzy set  $A$  in  $X$  is characterized by a membership function  $\mu_A : X \rightarrow [0,1]$ . Then  $A$  can be written as

$$A = \{(x, \mu_A(x)) | x \in X, 0 \leq \mu_A(x) \leq 1\}.$$

Definition of fuzzy point has been widely discussed by various authors, among others are as follows [9-13 and 15]:

**Definition 1** A Fuzzy point  $P_x$  in  $X$  is a fuzzy set whose membership function is

$$\mu_{P_x}(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

for all  $y \in X$ , where  $0 < \alpha < 1$ . We denote fuzzy points as  $x_\alpha$  or  $(x, \alpha)$ .

**Definition 2** Two fuzzy points  $x_\alpha$  and  $y_\beta$  is said equal if  $x = y$  and  $\alpha = \beta$  where  $\alpha, \beta \in (0, 1]$ .

**Definition 3** Let  $x_\alpha$  be a fuzzy point and  $A$  be a fuzzy set in  $X$ . Then  $x_\alpha$  is said belong to  $A$  or  $x_\alpha \in A$  if  $\alpha \leq \mu_A(x)$ .

Let  $X$  be a vector space over a field  $K$  and  $A$  be a fuzzy set in  $X$ . Then  $A$  is said to be a fuzzy subspace in  $X$  if for all  $x, y \in X$  and  $\lambda \in K$  satisfy

- (i)  $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$
- (ii)  $\mu_A(\lambda x) \geq \mu_A(x)$ .

Fuzzy normed space is defined in various ways. The definition based on an instituinistic approach is different from the fuzzy point approach. The following are given the definition of fuzzy normed space based on the fuzzy point approach as discussed on [15], and definition of fuzzy normed used in this paper is as given by [1] and [15].

**Definition 4** Let  $X$  be a vector space over a field  $K$ , and let  $\|\cdot\|_f : \tilde{A} \rightarrow [0, \infty)$  be a function that associates each point  $x_\alpha$  in  $\tilde{A}$ ,  $\alpha \in (0, 1]$  to nonnegative real number  $\|\cdot\|_f$  so that

(FN1)  $\|x_\alpha\|_f = 0$  if only if  $x = 0$ .

(FN2)  $\|\lambda x_\alpha\|_f = |\lambda| \|x_\alpha\|_f$  for all  $\lambda \in K$ .

$$(FN3) \quad \|x_\alpha + y_\beta\|_f = \|x_\alpha\|_f + \|y_\beta\|_f.$$

(FN4) If  $0 < \sigma \leq \alpha < 1$ , then  $\|x_\alpha\|_f \leq \|x_\sigma\|$  and there exists  $0 < \alpha_n \leq \alpha < 1$ , such that  $\lim_{n \rightarrow \infty} \|x_{\alpha_n}\|_f = \|x_\alpha\|_f$ . Then  $\|\cdot\|_f$  is called a fuzzy norm on  $X$  and a pair  $(X, \|\cdot\|_f)$  is called a fuzzy normed space.

**Definition 5** Let  $n \in \mathbb{N}$  dan  $X$  be a vector space over  $R$  with  $\dim(X) \geq n$ . Define function  $\|\cdot, \dots, \cdot\|: X \times X \times \dots \times X \rightarrow R$  with  $(x_1, x_2, \dots, x_n) \mapsto \|x_1, x_2, \dots, x_n\|$ .   
 *n times*

Function  $\|\cdot, \dots, \cdot\|$  is called  $n$ -norm on  $X$  if for every  $x_1, x_2, \dots, x_n, y, z \in X$  satisfying the conditions:

(nN1)  $\|x_1, x_2, \dots, x_n\| = 0$  if only if  $x_1, x_2, \dots, x_n$  linearly dependent.

(nN2)  $\|x_1, x_2, \dots, x_n\|$  invariant under permutation.

(nN3)  $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$  for any  $\alpha \in R$ .

(nN4)  $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ .

A pair  $(X; \|\cdot, \dots, \cdot\|)$  is called  $n$ -normed space.

From definition of  $n$ -norm and fuzzy norm on  $X$  above, based on the  $n$ -normed construction on [6], the author constructs a new function to produce a new space that called fuzzy  $n$ -normed as follows.

**Definition 6** Let  $n \in \mathbb{N}$  and  $X$  be a vector space over  $R$ . Define function

$$\|\cdot, \dots, \cdot\|_f: \tilde{A} \times \tilde{A} \times \dots \times \tilde{A} \rightarrow [0, \infty)$$

*n times*

where

$$(x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}) \mapsto \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f$$

with  $\alpha^{(i)} \in (0, 1]$  and  $x_{\alpha^{(i)}}^{(i)} \in \tilde{A}$  for  $i = 1, 2, \dots, n$ . Function  $\|\cdot, \dots, \cdot\|_f$  is called Fuzzy  $n$ -normed if satisfy the conditions :

(FnN1)  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = 0$  if only if  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$  linearly dependent.

(FnN2)  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f$  invariant under permutation.

(FnN3)  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, rx_{\alpha^{(n)}}^{(n)}\|_f = |r| \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f$  for  $\forall r \in R$ .

(FnN4)  $\|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_\beta + z_\gamma\|_f \leq \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_\beta\|_f + \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, z_\gamma\|_f$ .

(FnN5) If  $0 < \rho_i \leq \alpha^{(i)} < 1$  for  $i = 1, 2, \dots, n$  then

$$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f \leq \|x_{\rho_1}^{(1)}, x_{\rho_2}^{(2)}, \dots, x_{\rho_n}^{(n)}\|_f$$

and there exist  $0 < \alpha_m^{(i)} < \alpha^{(i)}$  for  $i = 1, 2, \dots, n$  and  $m \in N$  such that

$$\lim_{m \rightarrow \infty} \|x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n)}}^{(n)}\|_f = \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f.$$

A pair  $(X, \|\cdot, \dots, \cdot\|_f)$  is called Fuzzy  $n$ -Normed Space.

In [13], [12] and [10] it is proved that every normed space is fuzzy normed space. Conversely is also true. In other side, in [15] it is also mentioned that 2-normed space and fuzzy 2-normed space is equivalent. The following is given a theorem that ensures that any every  $n$ -normed spaces is fuzzy  $n$ -normed space.

**Theorem 7** Let  $n \in N$ ,  $X$  be a vector space over  $R$  and  $\dim(X) \geq n$ . Suppose  $(X; \|\cdot, \dots, \cdot\|)$  is  $n$ -normed space. Define

$$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\|$$

where  $\delta = \max\{\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(n)}\}$ . Then  $(X, \|\cdot, \dots, \cdot\|_f)$  is fuzzy  $n$ -normed space.

*Proof.* Suppose  $x_{\alpha^{(i)}}^{(i)}, y_\beta, z_\gamma \in \tilde{A}$ ,  $\alpha^{(i)} \in (0, 1]$  for  $i = 1, 2, \dots, n$  and  $r \in R$ . Then

$$(FnN1) \quad \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = 0 \Leftrightarrow \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| = 0$$

$$\Leftrightarrow \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| = 0$$

$$\Leftrightarrow x^{(1)}, x^{(2)}, \dots, x^{(n)} \text{ are linearly dependent.}$$

(FnN2) Since  $\|x^{(1)}, x^{(2)}, \dots, x^{(n)}\|$  is invariant under permutation, then

$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f$  is invariant under permutation.

$$\begin{aligned}
 (\text{FnN3}) \quad \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, rx_{\alpha^{(n)}}^{(n)}\|_f &= \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, rx^{(n)}\| \\
 &= \frac{1}{\delta} \|r\| \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\
 &= \|r\| \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\
 &= \|r\| \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f
 \end{aligned}$$

$$\begin{aligned}
 (\text{FnN4}) \quad \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta} + z_{\gamma}\|_f &= \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, (y+z)_{\tau}\| \quad \text{with } \tau = \max\{\beta, \gamma\} \\
 &= \frac{1}{\delta} \|x^{(1)}, \dots, x^{(n-1)}, y+z\| \quad \text{and } \delta = \max\{\alpha^{(1)}, \dots, \alpha^{(n-1)}, \tau\} \\
 &\leq \frac{1}{\delta} (\|x^{(1)}, \dots, x^{(n-1)}, y\| + \|x^{(1)}, \dots, x^{(n-1)}, z\|) \\
 &= \frac{1}{\delta} \|x^{(1)}, \dots, x^{(n-1)}, y\| + \frac{1}{\delta} \|x^{(1)}, \dots, x^{(n-1)}, z\| \\
 &\leq \frac{1}{\delta_1} \|x^{(1)}, \dots, x^{(n-1)}, y\| + \frac{1}{\delta_2} \|x^{(1)}, \dots, x^{(n-1)}, z\| \\
 &= \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta}\|_f + \|x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, z_{\gamma}\|_f
 \end{aligned}$$

where  $\delta_1 = \max\{\alpha^{(1)}, \dots, \alpha^{(n-1)}, \beta\}$  and  $\delta_2 = \max\{\alpha^{(1)}, \dots, \alpha^{(n-1)}, \gamma\}$

(FnN5) Suppose  $\eta = \max_{i \in \{1, \dots, n\}} \{\rho_i\}$  and  $\delta = \max_{i \in \{1, \dots, n\}} \{\alpha^{(i)}\}$ .

If  $0 < \rho_i \leq \alpha^{(i)} < 1$  for  $i = 1, 2, \dots, n$ , then  $\eta \leq \delta$ , such that  $\frac{1}{\delta} \leq \frac{1}{\eta}$ . Then

$$\begin{aligned}
 \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f &= \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\
 &\leq \frac{1}{\eta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\
 &= \|x_{\rho_1}^{(1)}, x_{\rho_2}^{(2)}, \dots, x_{\rho_n}^{(n)}\|_f.
 \end{aligned}$$

Further, suppose  $H_i = \{r \mid 0 < r < \alpha^{(i)}\}$  for  $i = 1, 2, \dots, n$ . Since  $H_i$  is bounded, then there is a sequence  $(\alpha_m^{(i)})$  in  $H_i$  for  $i = 1, 2, \dots, n$  that converges. Construct sequence  $(\alpha_m^{(i)})$  where  $\alpha_m^{(i)} = \frac{\alpha_{m-1}^{(i)} + \alpha^{(i)}}{2}$  for  $i = 1, 2, \dots, n$ . Then sequence  $(\alpha_m^{(i)})$

converges to  $\alpha^{(i)}$  for  $i=1,2,\dots,n$ . If  $\delta_m = \max_{i \in \{1,\dots,n\}} \{\alpha_m^{(i)}\}$  and  $\delta = \max_{i \in \{1,\dots,n\}} \{\alpha^{(i)}\}$ , then sequence  $(\delta_m)$  converges to  $\delta$  such that

$$\begin{aligned} \lim_{m \rightarrow \infty} \|x_{\alpha_m^{(1)}}, x_{\alpha_m^{(2)}}, \dots, x_{\alpha_m^{(n)}}\|_f &= \lim_{m \rightarrow \infty} \frac{1}{\delta_m} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\ &= \frac{1}{\delta} \|x^{(1)}, x^{(2)}, \dots, x^{(n)}\| \\ &= \|x_{\alpha^{(1)}}, x_{\alpha^{(2)}}, \dots, x_{\alpha^{(n)}}\|_f \quad \square \end{aligned}$$

In [6], it is described that every  $n$ -normed space is norm space. The following is given a theorem which states that fuzzy  $n$ -normed space is fuzzy normed space.

**Theorem 8** Let  $(X, \|\cdot, \dots, \cdot\|_f)$  be a fuzzy  $n$ -normed space for all  $n \geq 2$ . Define

$$\|x_\delta\|_f = \|x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f + \|x_\delta, a_{\alpha^{(n)}}^{(n)}\|_f$$

where  $a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n)}}^{(n)}$  are linearly independent vectors. Then  $(X, \|\cdot\|_f)$  is fuzzy normed space.

*Proof.* (FN1) If  $x_\delta = 0$  then obviously  $\|x_\delta\|_f = 0$ . If  $\|x_\delta\|_f = 0$  then  $\|x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f = \|x_\delta, a_{\alpha^{(n)}}^{(n)}\|_f = 0$ . Since  $\|x_\delta, a_{\alpha^{(n)}}^{(n)}\|_f = 0$ , then  $x_\delta, a_{\alpha^{(n)}}^{(n)}$  linearly dependent. But, since  $a_{\alpha^{(n)}}^{(n)}$  linearly independent, then  $x_\delta = t a_{\alpha^{(n)}}^{(n)}$  for some  $t \in R$ . Since  $\|x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f = 0$ , then

$$\begin{aligned} \|t a_{\alpha^{(n)}}^{(n)}, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f &= 0 \\ |t| \|a_{\alpha^{(n)}}^{(n)}, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f &= 0. \end{aligned}$$

In other side, since  $a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n)}}^{(n)}$  are linearly independent vectors. Then  $\|a_{\alpha^{(n)}}^{(n)}, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f > 0$  so must necessarily  $|t| = 0$  or  $t = 0$ . Consequently,  $x_\delta = t a_{\alpha^{(n)}}^{(n)} = 0 a_{\alpha^{(n)}}^{(n)} = 0$ .

$$(FN2) \quad \|\lambda x_\delta\|_f = \|\lambda x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)}\|_f + \|\lambda x_\delta, a_{\alpha^{(n)}}^{(n)}\|_f$$



$$\begin{aligned}
&= \lambda \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \lambda \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f \\
&= \lambda \left( \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f \right) \\
&= \lambda \left\| x_\delta \right\|_f.
\end{aligned}$$

$$(FN3) \quad \left\| x_\delta + y_\gamma \right\|_f = \left\| x_\delta + y_\gamma, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta + y_\gamma, a_{\alpha^{(n)}}^{(n)} \right\|_f$$

$$\begin{aligned}
&\leq \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| y_\gamma, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f + \left\| y_\gamma, a_{\alpha^{(n)}}^{(n)} \right\|_f \\
&= \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| y_\gamma, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f + \left\| y_\gamma, a_{\alpha^{(n)}}^{(n)} \right\|_f \\
&= \left\| x_\delta \right\|_f + \left\| y_\gamma \right\|_f
\end{aligned}$$

(FN4) If  $0 < \sigma \leq \delta < 1$  then use (FnN4),

$$\left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \leq \left\| x_\sigma, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \text{ and } \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f \leq \left\| x_\sigma, a_{\alpha^{(n)}}^{(n)} \right\|_f.$$

So that

$$\left\| x_\delta \right\|_f = \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f \leq \left\| x_\sigma, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\sigma, a_{\alpha^{(n)}}^{(n)} \right\|_f = \left\| x_\sigma \right\|_f.$$

Further, there exist  $0 < \alpha_m^{(i)} < \alpha^{(i)}$  for  $i=1, 2, \dots, n$  and  $m \in N$  so using (FnN5),

$$\lim_{m \rightarrow \infty} \left\| x_{\delta_m}, a_{\alpha_m^{(1)}}^{(1)}, \dots, a_{\alpha_m^{(n-1)}}^{(n-1)} \right\|_f = \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \text{ and } \lim_{m \rightarrow \infty} \left\| x_{\delta_m}, a_{\alpha_m^{(n)}}^{(n)} \right\|_f = \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f.$$

So that

$$\begin{aligned}
\lim_{m \rightarrow \infty} \left\| x_{\delta_m} \right\|_f &= \lim_{m \rightarrow \infty} \left( \left\| x_{\delta_m}, a_{\alpha_m^{(1)}}^{(1)}, \dots, a_{\alpha_m^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_{\delta_m}, a_{\alpha_m^{(n)}}^{(n)} \right\|_f \right) \\
&= \lim_{m \rightarrow \infty} \left\| x_{\delta_m}, a_{\alpha_m^{(1)}}^{(1)}, \dots, a_{\alpha_m^{(n-1)}}^{(n-1)} \right\|_f + \lim_{m \rightarrow \infty} \left\| x_{\delta_m}, a_{\alpha_m^{(n)}}^{(n)} \right\|_f \\
&= \left\| x_\delta, a_{\alpha^{(1)}}^{(1)}, \dots, a_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| x_\delta, a_{\alpha^{(n)}}^{(n)} \right\|_f \\
&= \left\| x_\delta \right\|_f.
\end{aligned}$$

□

In [6], it be proved that every  $n$ -normed space is  $(n-1)$  normed space. The following, we will proof a theorem which states that every fuzzy  $n$ -normed space is fuzzy  $(n-1)$  normed space.

**Theorem 9** Let  $(X, \|\cdot, \dots, \cdot\|_f)$  be fuzzy  $n$ -normed space for all  $n \geq 2$ . Define

$$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}\|_f^\infty = \max_{i \in \{1, \dots, n\}} \left\{ \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}\|_f \right\}$$

where  $a_{\beta^{(1)}}^{(1)}, \dots, a_{\beta^{(n)}}^{(n)}$  are linearly independent vectors. Then  $(X, \|\cdot, \dots, \cdot\|_f^\infty)$  is fuzzy  $(n-1)$ -normed space.

*Proof.*

(1) If  $x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}$  are linearly dependent, then  $x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}$  for

$i = 1, \dots, n$  are linearly dependent too so that  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}\|_f = 0$  for

$i = 1, \dots, n$ . As a result,  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}\|_f^\infty = 0$ . Conversely, if

$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}\|_f^\infty = 0$ , then  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}\|_f = 0$  for  $i = 1, \dots, n$ .

Then  $x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}$  for  $i = 1, \dots, n$  are linearly dependent. Since

$a_{\beta^{(1)}}^{(1)}, \dots, a_{\beta^{(n)}}^{(n)}$  are linearly independent vectors, then we must have

$x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}$  linearly dependent.

(2) Since  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}\|_f$  is invariant under permutations, then

$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}\|_f^\infty$  is invariant under permutations too.

$$\begin{aligned} (3) \quad \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, rx_{\alpha^{(n-1)}}^{(n-1)}\|_f^\infty &= \max_{i \in \{1, \dots, n\}} \left\{ \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, rx_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)}\|_f \right\} \\ &= \max_{i \in \{1, \dots, n\}} \left\{ r \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \right\} \\ &= r \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \\ &= r \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \text{ for all } r \in R. \end{aligned}$$

$$\begin{aligned} (4) \quad \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, y_\xi + z_\omega\|_f^\infty &= \max_{i \in \{1, \dots, n\}} \left\{ \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, y_\xi + z_\omega, a_{\beta^{(i)}}^{(i)}\|_f \right\} \\ &\leq \max_{i \in \{1, \dots, n\}} \left\{ \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, y_\xi, a_{\beta^{(i)}}^{(i)}\|_f + \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, z_\omega, a_{\beta^{(i)}}^{(i)}\|_f \right\} \\ &= \max_{i \in \{1, \dots, n\}} \left\{ \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, y_\xi, a_{\beta^{(i)}}^{(i)}\|_f + \max_{i \in \{1, \dots, n\}} \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, z_\omega, a_{\beta^{(i)}}^{(i)}\|_f \right\} \\ &= \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, y_\xi\|_f^\infty + \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-2)}}^{(n-2)}, z_\omega\|_f^\infty \end{aligned}$$

(5) If  $0 < \rho_i \leq \alpha^{(i)} < 1$  for  $i = 1, 2, \dots, n-1$  then

$$\begin{aligned} \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f^\infty &= \max_{i \in \{1, \dots, n\}} \left\{ \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \right\} \\ &\leq \max_{i \in \{1, \dots, n\}} \left\{ \left\| x_{\rho_1}^{(1)}, x_{\rho_2}^{(2)}, \dots, x_{\rho_{n-1}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \right\} \\ &= \left\| x_{\rho_1}^{(1)}, x_{\rho_2}^{(2)}, \dots, x_{\rho_{n-1}}^{(n-1)} \right\|_f^\infty \end{aligned}$$

and there exist  $0 < \alpha_m^{(i)} < \alpha^{(i)}$  for  $i = 1, 2, \dots, n-1$  and  $0 < \beta_m^{(i)} < \beta^{(i)}$  for  $i = 1, 2, \dots, n$  and  $m \in \mathbb{N}$  such that

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\| x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n-1)}}^{(n-1)} \right\|_f^\infty &= \lim_{m \rightarrow \infty} \max_{i \in \{1, \dots, n\}} \left\{ \left\| x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n-1)}}^{(n-1)}, a_{\beta_m^{(i)}}^{(i)} \right\|_f \right\} \\ &= \max_{i \in \{1, \dots, n\}} \left\{ \lim_{m \rightarrow \infty} \left\| x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n-1)}}^{(n-1)}, a_{\beta_m^{(i)}}^{(i)} \right\|_f \right\} \\ &= \max_{i \in \{1, \dots, n\}} \left\{ \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, a_{\beta^{(i)}}^{(i)} \right\|_f \right\} \\ &= \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f^\infty \quad \square \end{aligned}$$

**Corollary 10** Every fuzzy  $n$ -normed space is a fuzzy  $(n-r)$ -normed space for  $r = 1, \dots, n-1$ . In particular, every fuzzy  $n$ -normed space is a fuzzy normed space.

*Proof.* This is a direct result of Theorem 9. □

**Corollary 11** Every  $n$ -normed space is a fuzzy normed space.

*Proof.* Apply Theorem 7 dan 8. □

Whereas the relationship between the normed space and the fuzzy normed space and vice versa has been discussed in [5] and [6].

**Theorem 12** Let  $(X, \|\cdot\|)$  be a normed space. Define  $\|x_\alpha\|_f = \frac{1}{\alpha} \|x\|$  for all  $x_\alpha \in X$  where  $\alpha \in (0, 1]$ . Then  $(X, \|\cdot\|_f)$  is a fuzzy normed space.

**Theorem 13** Let  $(X, \|\cdot\|_f)$  be fuzzy normed space. Define  $\|x\| = \|x_1\|_f = \|(x, 1)\|_f$  for all  $x \in X$ . Then  $(X, \|\cdot\|)$  is a norm space.

**Corollary 14** Every fuzzy  $n$ -normed space is a norm space.

*Proof.* Apply Corollary 10 dan Theorem 13. □

The following is given the definition of  $n$ -inner product that is different from the definition in [7] and [2].

**Definition 15** Let  $n \in \mathbb{N}$  and  $n \geq 2$ ,  $X$  be a vector space over  $R$  and  $\dim(X) \geq n$ . Function  $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : X \times X \times \cdots \times X \rightarrow R$   $n+1$  times is called  $n$ -inner product on  $X$  if for every  $x_1, x_2, \dots, x_n, y, z \in X$  satisfy the conditions :

$$(nIP1) \quad \langle x_1, x_1 | x_2, \dots, x_n \rangle \geq 0 \quad \text{and}$$

$$\langle x_1, x_1 | x_2, \dots, x_n \rangle = 0 \quad \text{if only if} \quad x_1, x_2, \dots, x_n \text{ linearly dependent.}$$

$$(nIP2) \quad \langle x_1, y | x_2, \dots, x_n \rangle = \langle y, x_1 | x_2, \dots, x_n \rangle.$$

$$(nIP3) \quad \langle x_1, y | x_2, \dots, x_n \rangle \text{ invariant under permutation } x_2, \dots, x_n.$$

$$(nIP4) \quad \langle x_1, x_1 | x_2, \dots, x_n \rangle = \langle x_2, x_2 | x_1, x_3, \dots, x_n \rangle.$$

$$(nIP5) \quad \langle \alpha x_1, x_1 | x_2, \dots, x_n \rangle = \alpha \langle x_1, x_1 | x_2, \dots, x_n \rangle \text{ for } \forall \alpha \in R.$$

$$(nIP6) \quad \langle y + z, x_1 | x_2, \dots, x_n \rangle = \langle y, x_1 | x_2, \dots, x_n \rangle + \langle z, x_1 | x_2, \dots, x_n \rangle.$$

A pair  $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$  is called  $n$ -inner product space.

**Definition 16** Let  $X$  be a vector space over  $K = \mathbb{C}$  atau  $K = \mathbb{R}$ . Define function

$\langle \cdot, \cdot \rangle_f : \tilde{A} \times \tilde{A} \rightarrow K$  with  $\langle x_\alpha, y_\beta \rangle_f = \langle x, y \rangle(\lambda)$  for every  $x_\alpha, y_\beta \in \tilde{A}$ , where  $\lambda = \min\{\alpha, \beta\}$ , and  $\alpha, \beta \in (0, 1]$ . For every  $x_\alpha, y_\beta, z_\gamma \in \tilde{A}$ ,  $\lambda = \min\{\alpha, \beta, \gamma\}$  and  $\alpha, \beta, \gamma \in (0, 1]$ . Function  $\langle \cdot, \cdot \rangle_f$  is called fuzzy inner product if it satisfies the conditions:

(FIP1)  $\langle x, x \rangle(\lambda) \geq 0$  and  $\langle x, x \rangle(\lambda) = 0$  if only if  $x = 0$ .

(FIP2)  $\langle x, y \rangle(\lambda) = \overline{\langle y, x \rangle(\lambda)}$ .

(FIP3)  $\langle rx, y \rangle(\lambda) = r \langle x, y \rangle(\lambda)$  for  $\forall r \in K$ .

(FIP4)  $\langle x + y, z \rangle(\lambda) = \langle x, z \rangle(\lambda) + \langle y, z \rangle(\lambda)$ .

(FIP5) If  $0 < \sigma \leq \alpha < 1$  then  $\langle x, x \rangle(\alpha) \leq \langle x, x \rangle(\sigma)$ , and there exist  $0 < \alpha_n \leq \alpha$  so that  $\lim_{n \rightarrow \infty} \langle x, x \rangle(\alpha_n) = \langle x, x \rangle(\alpha)$ .

A pair  $(X, \langle \cdot, \cdot \rangle_f)$  is called fuzzy inner product space.

From definition of  $n$ -inner product and fuzzy inner product on  $X$  above, can be constructed a new function called fuzzy  $n$ -inner product as follows.

**Definition 17** Let  $n \in N$  and  $n \geq 2$ ,  $X$  be a vector space over  $R$  and  $\dim(X) \geq n$ . Define function

$$\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : \tilde{A} \times \tilde{A} \times \dots \times \tilde{A} \rightarrow [0, \infty)$$

$n+1$  times

with

$$\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} | x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \rangle_f = \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda)$$

where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}\}$ ,  $\alpha^{(i)} \in (0, 1]$  and  $(x^{(i)})_{\alpha^{(i)}} \in \tilde{A}$  for  $i = 1, 2, \dots, n$ . Function  $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$  is called fuzzy  $n$ -inner product if it satisfies the conditions :

(FnIP1)  $\langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) \geq 0$  for  $\forall x^{(1)}, x^{(2)}, \dots, x^{(n)} \in P(X)$  and

$\langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = 0$  if only if  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$  are linearly dependent.

(FnIP2)  $\langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \langle y, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda)$  where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}, \beta\}$ .

(FnIP3)  $\langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda)$  invariant under permutation where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}, \beta\}$ .

(FnIP4)  $\langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \langle x^{(2)}, x^{(2)} | x^{(1)}, x^{(3)}, \dots, x^{(n)} \rangle(\lambda)$ .

$$(FnIP5) \quad \langle rx^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = r \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) \text{ for } \forall r \in R.$$

$$(FnIP6) \quad \langle x^{(1)} + z, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) + \langle z, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda).$$

where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}, \beta, \gamma\}.$

(FnIP7) If  $0 < \sigma \leq \rho < 1$  then

$$\langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho) \leq \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\sigma)$$

and there exist  $0 < \rho_m \leq \rho$  so that

$$\lim_{m \rightarrow \infty} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho_m) = \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho).$$

A pair  $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_f)$  is called fuzzy  $n$ -inner product space.

In [11] it is proved that every inner product space is fuzzy inner product space. Conversely is also true. The following is provided theorem that ensure that every  $n$ -inner product space is fuzzy  $n$ -inner product space.

**Theorem 18** Let  $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$  be a  $n$ -inner product space. Define

$$\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} | x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(1)}}^{(1)} \rangle_f = \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \frac{1}{\lambda} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle.$$

Then  $(X; \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_f)$  is a fuzzy  $n$ -inner product space.

$$Proof. \quad (FnIP1) \quad \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \frac{1}{\lambda} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \geq 0.$$

$$\begin{aligned} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = 0 &\Leftrightarrow \frac{1}{\lambda} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle = 0 \\ &\Leftrightarrow \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle = 0 \\ &\Leftrightarrow x^{(1)} x^{(2)}, \dots, x^{(n)} \quad \text{linearly dependent.} \end{aligned}$$

$$(FnIP2) \quad \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle(\lambda) = \frac{1}{\lambda} \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle$$

$$\begin{aligned}
&= \frac{1}{\lambda} \langle y, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \langle y, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle (\lambda).
\end{aligned}$$

(FnIP3) Applying (nIP3), that is  $\langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle$  is invariant under permutation  $x^{(2)}, \dots, x^{(n)}$ . Then  $\langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle (\lambda) = \frac{1}{\lambda} \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle$  invariant under permutation where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}, \beta\}$ .

$$\begin{aligned}
\text{(FnIP4)} \quad \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle (\lambda) &= \frac{1}{\lambda} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \frac{1}{\lambda} \langle x^{(2)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \langle x^{(2)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \lambda
\end{aligned}$$

$$\begin{aligned}
\text{(FnIP5)} \quad \langle rx^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle (\lambda) &= \frac{1}{\lambda} \langle rx^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \frac{1}{\lambda} r \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= r \frac{1}{\lambda} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \\
&= r \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \lambda \quad \text{for } \lambda \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
\text{(FnIP6)} \quad \langle x^{(1)} + z, y | x^{(2)}, \dots, x^{(n)} \rangle (\lambda) &= \frac{1}{\lambda} \langle x^{(1)} + z, y | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \frac{1}{\lambda} \left( \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle + \langle z, y | x^{(2)}, \dots, x^{(n)} \rangle \right) \\
&= \frac{1}{\lambda} \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle + \frac{1}{\lambda} \langle z, y | x^{(2)}, \dots, x^{(n)} \rangle \\
&= \langle x^{(1)}, y | x^{(2)}, \dots, x^{(n)} \rangle (\lambda) + \langle z, y | x^{(2)}, \dots, x^{(n)} \rangle (\lambda)
\end{aligned}$$

where  $\lambda = \min_{i \in \{1, \dots, n\}} \{\alpha^{(i)}, \beta, \gamma\}$ .

(FnIP7) If  $0 < \sigma \leq \rho < 1$  then  $\frac{1}{\rho} \leq \frac{1}{\sigma}$ . We get

$$\frac{1}{\rho} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle \leq \frac{1}{\rho} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle$$

$$\langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho) \leq \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\sigma)$$

and there exists  $0 < \rho_m \leq \rho$  with  $\rho_m$  converges to  $\rho$ . So that

$$\lim_{m \rightarrow \infty} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho_m) = \lim_{m \rightarrow \infty} \frac{1}{\rho_m} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle$$

$$= \frac{1}{\rho} \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle$$

$$= \langle x^{(1)}, x^{(1)} | x^{(2)}, \dots, x^{(n)} \rangle(\rho). \quad \square$$

In [11] it is proved that every fuzzy inner product space is fuzzy normed space. However, conversely is not true. For example, one can see in [9]. In other side, in [15] it proved that fuzzy 2-inner product space is fuzzy 2-normed space, but for general case has not yet provided. The following is given a theorem which ensures that every fuzzy  $n$ -inner product space is a fuzzy  $n$ -normed space.

**Theorem 19** Let  $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_f)$  be a fuzzy  $n$ -inner product space for  $n \geq 1$ .

Define

$$\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = \langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} | x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \rangle_f^{\frac{1}{2}}$$

with  $\alpha^{(i)} \in (0, 1]$  and  $x_{\alpha^{(i)}}^{(i)} \in \tilde{A}$  for  $i = 1, 2, \dots, n$ . then  $(X; \|\cdot\|, \dots, \|\cdot\|_f)$  is a fuzzy  $n$ -normed space.

*Proof.* (FnN1)  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = 0 \Leftrightarrow \langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} | x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \rangle_f = 0$

$$\Leftrightarrow x^{(1)}, x^{(2)}, \dots, x^{(n)} \text{ linearly dependent.}$$

(FnN2)  $\|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f$  invariant under permutation.

(FnN3)  $\|rx_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f = \langle rx_{\alpha^{(1)}}^{(1)}, rx_{\alpha^{(1)}}^{(1)} | x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \rangle_f^{\frac{1}{2}}$



$$\begin{aligned}
&= \left( r \left\langle x_{\alpha^{(1)}}^{(1)}, rx_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f \right)^{\frac{1}{2}} \\
&= \left( r \left\langle rx_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f \right)^{\frac{1}{2}} \\
&= \left( r^2 \left\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f \right)^{\frac{1}{2}} \\
&= |r| \left\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f^{\frac{1}{2}} \\
&= |r| \left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\|_f \text{ for all } r \in R.
\end{aligned}$$

$$\begin{aligned}
(\text{FnN4}) \quad & \left( \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta} + z_{\gamma} \right\|_f \right)^2 = \left( \left\| y_{\beta} + z_{\gamma}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \right)^2 \\
&= \left\langle y_{\beta} + z_{\gamma}, y_{\beta} + z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&= \left\langle y_{\beta}, y_{\beta} + z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + \left\langle z_{\gamma}, y_{\beta} + z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&= \left\langle y_{\beta} + z_{\gamma}, y_{\beta} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + \left\langle y_{\beta} + z_{\gamma}, z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&= \left\langle y_{\beta}, y_{\beta} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + \left\langle z_{\gamma}, y_{\beta} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + \left\langle y_{\beta}, z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&\quad + \left\langle z_{\gamma}, z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&= \left\langle y_{\beta}, y_{\beta} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + 2 \left\langle y_{\beta}, z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f + \left\langle z_{\gamma}, z_{\gamma} \mid x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\rangle_f \\
&\leq \left\| y_{\beta}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f^2 + 2 \left\| y_{\beta}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \left\| z_{\gamma}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| z_{\gamma}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f^2 \\
&= \left( \left\| y_{\beta}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f + \left\| z_{\gamma}, x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)} \right\|_f \right)^2 \\
&= \left( \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta} \right\|_f + \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, z_{\gamma} \right\|_f \right)^2.
\end{aligned}$$

so that

$$\left( \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta} + z_{\gamma} \right\|_f \right) \leq \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, y_{\beta} \right\|_f + \left\| x_{\alpha^{(1)}}^{(1)}, \dots, x_{\alpha^{(n-1)}}^{(n-1)}, z_{\gamma} \right\|_f.$$

(FnN5) If  $0 < \rho_i \leq \alpha^{(i)} < 1$  for  $i = 1, 2, \dots, n$  then

$$\begin{aligned}
\left\| x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\|_f &= \left\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f^{\frac{1}{2}} \\
&\leq \left\langle x_{\rho_1}^{(1)}, x_{\rho_1}^{(1)} \mid x_{\rho_2}^{(2)}, \dots, x_{\rho_n}^{(n)} \right\rangle_f^{\frac{1}{2}} \\
&= \left\| x_{\rho_1}^{(1)}, x_{\rho_2}^{(2)}, \dots, x_{\rho_n}^{(n)} \right\|_f.
\end{aligned}$$

and there exist  $0 < \alpha_m^{(i)} < \alpha^{(i)}$  for  $i = 1, 2, \dots, n$  and  $m \in N$  so that

$$\begin{aligned} \lim_{m \rightarrow \infty} \|x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n)}}^{(n)}\|_f &= \lim_{m \rightarrow \infty} \left\langle x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(1)}}^{(1)} \mid x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n)}}^{(n)} \right\rangle_f^{\frac{1}{2}} \\ &= \left( \lim_{m \rightarrow \infty} \left\langle x_{\alpha_m^{(1)}}^{(1)}, x_{\alpha_m^{(1)}}^{(1)} \mid x_{\alpha_m^{(2)}}^{(2)}, \dots, x_{\alpha_m^{(n)}}^{(n)} \right\rangle_f \right)^{\frac{1}{2}} \\ &= \left\langle x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(1)}}^{(1)} \mid x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)} \right\rangle_f^{\frac{1}{2}} \\ &= \|x_{\alpha^{(1)}}^{(1)}, x_{\alpha^{(2)}}^{(2)}, \dots, x_{\alpha^{(n)}}^{(n)}\|_f \quad \square \end{aligned}$$

**Corollary 20** Every  $n$ -inner product space is fuzzy  $n$ -normed space.

*Proof.* Apply Theorem 18 and 19.

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